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Abstract

Using recent results on the behavior of multiple Wiener-Itô integrals based on Stein's method, we prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables related to the increments of the fractional Brownian motion.

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1 Introduction

A famous result by Hsu and Robbins [7] says that if $X_1, X_2, \dots$ is a sequence of independent identically distributed random variables with zero mean and finite variance and $S_n := X_1 + \dots + X_n$, then

$$\sum_{n\geq 1} P\left(|S_n| > \varepsilon n\right) < \infty$$

for every $\varepsilon > 0$. Later, Erdös ([3], [4]) showed that the converse implication also holds, namely if the above series is finite for every $\varepsilon > 0$ and $X_1, X_2, \dots$ are independent and identically distributed, then $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$. Since then, many authors extended this result in several directions.

Spitzer's showed in [13] that

$$\sum_{n\geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right) < \infty$$

for every $\varepsilon > 0$ if and only if $\mathbf{E}X_1 = 0$ and $\mathbf{E}|X_1| < \infty$. Also, Spitzer's theorem has been the object of various generalizations and variants. One of the problems related to the Hsu-Robbins' and Spitzer's theorems is to find the precise asymptotic as $\varepsilon \to 0$ of the quantities

$\sum_{n\geq 1} P\left(|S_n|>\varepsilon n\right)$ and $\sum_{n\geq 1}\frac{1}{n}P\left(|S_n|>\varepsilon n\right)$. Heyde [5] showed that

$$\lim_{\varepsilon\to 0}\varepsilon^2\sum_{n\geq 1}P\left(|S_n|>\varepsilon n\right)=\mathbf{E}X_1^2 \tag{1}$$

whenever $\mathbf{E}X_1=0$ and $\mathbf{E}X_1^2<\infty$. In the case when X is attracted to a stable distribution of exponent $\alpha>1$, Spataru [12] proved that

$$\lim_{\varepsilon\to 0}\frac{1}{-\log\varepsilon}\sum_{n\geq 1}\frac{1}{n}P\left(|S_n|>\varepsilon n\right)=\frac{\alpha}{\alpha-1}. \tag{2}$$

The purpose of this paper is to prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables, related to the increments of fractional Brownian motion, in the spirit of [5] or [12]. Recall that the fractional Brownian motion $(B_t^H)_{t\in[0,1]}$ is a centered Gaussian process with covariance function $R^H(t,s)=\mathbf{E}(B_t^HB_s^H)=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$. It can be also defined as the unique self-similar Gaussian process with stationary increments. Concretely, in this paper we will study the behavior of the tail probabilities of the sequence

$$V_n=\sum_{k=0}^{n-1}H_q\left(n^H\left(B_{\frac{k+1}{n}}-B_{\frac{k}{n}}\right)\right) \tag{3}$$

where B is a fractional Brownian motion with Hurst parameter $H\in(0,1)$ (in the sequel we will omit the superscript H for B) and H_q is the Hermite polynomial of degree $q\geq 1$ given by $H_q(x)=(-1)^q e^{\frac{x^2}{2}}\frac{d^q}{dx^q}(e^{-\frac{x^2}{2}})$. The sequence V_n behaves as follows (see e.g. [9], Theorem 1; the result is also recalled in Section 3 of our paper): if $0<H<1-\frac{1}{2q}$, a central limit theorem holds for the renormalized sequence $Z_n^{(1)}=\frac{V_n}{c_{1,q,H}\sqrt{n}}$ while if $1-\frac{1}{2q}<H<1$, the sequence $Z_n^{(2)}=\frac{V_n}{c_{2,q,H}n^{1-q(1-H)}}$ converges in $L^2(\Omega)$ to a Hermite random variable of order q (see Section 2 for the definition of the Hermite random variable and Section 3 for a rigorous statement concerning the convergence of V_n). Here $c_{1,q,H},c_{2,q,H}$ are explicit positive constants depending on q and H.

We note that the techniques generally used in the literature to prove the Hsu-Robbins and Spitzer's results are strongly related to the independence of the random variables $X_1,X_2,\ldots$. In our case the variables are correlated. Indeed, for any $k,l\geq 1$ we have $\mathbf{E}\left(H_q(B_{k+1}-B_k)H_q(B_{l+1}-B_l)\right)=\frac{1}{(q!)^2}\rho_H(k-l)$ where the correlation function is $\rho_H(k)=\frac{1}{2}\left((k+1)^{2H}+(k-1)^{2H}-2k^{2H}\right)$ which is not equal to zero unless $H=\frac{1}{2}$ (which is the case of the standard Brownian motion). We use new techniques based on the estimates for the multiple Wiener-Itô integrals obtained in [2], [10] via Stein's method and Malliavin calculus. Concretely, we study in this paper the behavior as $\varepsilon\to 0$ of the quantities

$$\sum_{n\geq 1}\frac{1}{n}P\left(V_n>\varepsilon n\right)=\sum_{n\geq 1}\frac{1}{n}P\left(Z_n^{(1)}>c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right), \tag{4}$$

and
$$\sum_{n\geq 1} P\left(V_n > \varepsilon n\right) = \sum_{n\geq 1} P\left(Z_n^{(1)} > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right), \tag{5}$$

if $0 < H < 1 - \frac{1}{2q}$ and of

$$\sum_{n\geq 1} \frac{1}{n} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n\geq 1} \frac{1}{n} P\left(Z_n^{(2)} > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right) \tag{6}$$

and
$$\sum_{n\geq 1} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n\geq 1} P\left(Z_n^{(2)} > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right) \tag{7}$$

if $1 - \frac{1}{2q} < H < 1$. The basic idea in our proofs is that, if we replace $Z_n^{(1)}$ and $Z_n^{(2)}$ by their limits (standard normal random variable or Hermite random variable) in the above expressions, the behavior as $\varepsilon \to 0$ can be obtained by standard calculations. Then we need to estimate the difference between the tail probabilities of $Z_n^{(1)}, Z_n^{(2)}$ and the tail probabilities of their limits. To this end, we will use the estimates obtained in [2], [10] via Malliavin calculus and we are able to prove that this difference converges to zero in all cases. We obtain that, as $\varepsilon \to 0$, the quantities (4) and (6) are of order of $\log \varepsilon$ while the functions (5) and (7) are of order of ε^2 and $\varepsilon^{1-q(1-H)}$ respectively.

The paper is organized as follows. Section 2 contains some preliminaries on the stochastic analysis on Wiener chaos. In Section 3 we prove the Spitzer's theorem for the variations of the fractional Brownian motion while Section 4 is devoted to the Hsu-Robbins theorem for this sequence.

Throughout the paper we will denote by c a generic strictly positive constant which may vary from line to line (and even on the same line).

2 Preliminaries

Let $(W_t)_{t\in[0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0,1]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W. The basic reference is [11].

Let $f \in \mathcal{S}_m$ be an elementary function with m variables that can be written as

$$f = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} 1_{A_{i_1}\times\ldots\times A_{i_m}}$$

where the coefficients satisfy $c_{i_1,\ldots i_m} = 0$ if two indices i_k and i_l are equal and the sets $A_i \in \mathcal{B}([0,1])$ are disjoint. For such a step function f we define

$$I_m(f) = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} W(A_{i_1})\ldots W(A_{i_m})$$

where we put $W(A) = \int_0^1 1_A(s)dW_s$ if $A \in \mathcal{B}([0,1])$. It can be seen that the mapping I_n constructed above from $\mathcal{S}_m$ to $L^2(\Omega)$ is an isometry on $\mathcal{S}_m$, i.e.

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = n!\langle f,g\rangle_{L^2([0,1]^n)} \text{ if } m=n \tag{8}$$

and

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = 0 \text{ if } m \neq n.$$

Since the set $\mathcal{S}_n$ is dense in $L^2([0,1]^n)$ for every $n \geq 1$ the mapping I_n can be extended to an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension.

We will need the following bound for the tail probabilities of multiple Wiener-Itô integrals (see [8], Theorem 4.1)

$$P\left(|I_n(f)| > u\right) \leq c \exp\left(\left(\frac{-cu}{\sigma}\right)^{\frac{2}{n}}\right) \tag{9}$$

for all $u > 0$, $n \geq 1$, with $\sigma = \|f\|_{L^2([0,1]^n)}$.

The Hermite random variable of order $q \geq 1$ that appears as limit in Theorem 1, point ii. is defined as (see [9])

$$Z = d(q,H)I_q(L) \tag{10}$$

where the kernel $L \in L^2([0,1]^q)$ is given by

$$L(y_1,\ldots,y_q) = \int_{y_1\vee\ldots\vee y_q}^1 \partial_1 K^H(u,y_1)\ldots\partial_1 K^H(u,y_q)du.$$

The constant $d(q,H)$ is a positive normalizing constant that guarantees that $\mathbf{E}Z^2 = 1$ and K^H is the standard kernel of the fractional Brownian motion (see [11], Section 5). We will not need the explicit expression of this kernel. Note that the case $q = 1$ corresponds to the fractional Brownian motion and the case $q = 2$ corresponds to the Rosenblatt process.

3 Spitzer's theorem

Let us start by recalling the following result on the convergence of the sequence V_n (3) (see [9], Theorem 1).

Theorem 1 *Let $q \geq 2$ an integer and let $(B_t)_{t\geq 0}$ a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then, with some explicit positive constants $c_{1,q,H}, c_{2,q,H}$ depending only on q and H we have*

i. If $0 < H < 1 - \frac{1}{2q}$ then

$$\frac{V_n}{c_{1,q,H}\sqrt{n}} \xrightarrow{\text{Law}}_{n\to\infty} N(0,1) \tag{11}$$

ii. *If* $1-\frac{1}{2q}<H<1$ *then*

$$\frac{V_n}{c_{2,q,H}n^{1-q(1-H)}} \xrightarrow{\mathrm{L}^2}{}_{n\to\infty} Z \tag{12}$$

where Z *is a Hermite random variable given by (10).*

In the case $H=1-\frac{1}{2q}$ the limit is still Gaussian but the normalization is different. However we will not treat this case in the present work.

We set

$$Z_n^{(1)}=\frac{V_n}{c_{1,q,H}\sqrt{n}},\quad Z_n^{(2)}=\frac{V_n}{c_{2,q,H}n^{1-q(1-H)}} \tag{13}$$

with the constants $c_{1,q,H}, c_{2,q,H}$ from Theorem 1.

Let us denote, for every $\varepsilon>0$,

$$f_1(\varepsilon)=\sum_{n\geq 1}\frac{1}{n}P\left(V_n>\varepsilon n\right)=\sum_{n\geq 1}\frac{1}{n}P\left(Z_n^{(1)}>c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) \tag{14}$$

and

$$f_2(\varepsilon)=\sum_{n\geq 1}\frac{1}{n}P\left(V_n>\varepsilon n^{2-2q(1-H)}\right)=\sum_{n\geq 1}\frac{1}{n}P\left(Z_n^{(2)}>c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right) \tag{15}$$

Remark 1 *It is natural to consider the tail probability of order* $n^{2-2q(1-H)}$ *in (15) because the* L^2 *norm of the sequence* V_n *is in this case of order* $n^{1-q(1-H)}$.

We are interested to study the behavior of $f_i(\varepsilon)$ $(i=1,2)$ as $\varepsilon\to 0$. For a given random variable X, we set $\Phi_X(z)=1-P(X<z)+P(X<-z)$.

The first lemma gives the asymptotics of the functions $f_i(\epsilon)$ as $\varepsilon\to 0$ when $Z_n^{(i)}$ are replaced by their limits.

Lemma 1 *Consider* $c>0$.

i. Let $Z^{(1)}$ *be a standard normal random variable. Then as*

$$\frac{1}{-\log c\varepsilon}\sum_{n\geq 1}\frac{1}{n}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{n})\to_{\varepsilon\to 0} 2.$$

ii. *Let* $Z^{(2)}$ *be a Hermite random variable or order* q *given by (10). Then, for any integer* $q\geq 1$

$$\frac{1}{-\log c\varepsilon}\sum_{n\geq 1}\frac{1}{n}\Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)})\to_{\varepsilon\to 0}\frac{1}{1-q(1-H)}.$$

Proof: The case when $Z^{(1)}$ follows the standard normal law is hidden in [12]. We will give the ideas of the proof. We can write (see [12])

$$\sum_{n\geq 1}\frac{1}{n}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) = \int_1^\infty \frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx - \frac{1}{2}\Phi_{Z^{(1)}}(c\varepsilon) - \int_1^\infty P_1(x)d\left[\frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})\right].$$

with $P_1(x) = [x] - x + \frac{1}{2}$. Clearly as $\varepsilon \to 0$, $\frac{1}{\log\varepsilon}\Phi_{Z^{(1)}}(c\varepsilon) \to 0$ because $\Phi_{Z^{(1)}}$ is a bounded function and concerning the last term it is also trivial to observe that

$$\begin{aligned}&\frac{1}{-\log c\varepsilon}\int_1^\infty P_1(x)d\left[\frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})\right]\\ &= \frac{1}{-\log c\varepsilon}\left(-\int_1^\infty P_1(x)\left(\frac{1}{x^2}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx + c\varepsilon\frac{1}{2}x^{-\frac{1}{2}}\frac{1}{x}\Phi'_{Z^{(1)}}(\varepsilon\sqrt{x})\right)dx\right) \to_{\varepsilon\to 0} 0\end{aligned}$$

since $\Phi_{Z^{(1)}}$ and $\Phi'_{Z^{(1)}}$ are bounded. Therefore the asymptotics of the function $f_1(\varepsilon)$ as $\varepsilon \to 0$ will be given by $\int_1^\infty \frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx$. By making the change of variables $c\varepsilon\sqrt{x} = y$, we get

$$\lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}\int_1^\infty \frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx = \lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}2\int_{c\varepsilon}^\infty \frac{1}{y}\Phi_{Z^{(1)}}(y)dy = \lim_{\varepsilon\to 0}2\Phi_{Z^{(1)}}(c\varepsilon) = 2.$$

Let us consider now the case of the Hermite random variable. We will have as above

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}\sum_{n\geq 1}\frac{1}{n}\Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)})\\ =\ &\lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}\left(\int_1^\infty \frac{1}{x}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx - \int_1^\infty P_1(x)d\left[\frac{1}{x}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]\right)\end{aligned}$$

By making the change of variables $c\varepsilon x^{1-q(1-H)} = y$ we will obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}\int_1^\infty \frac{1}{x}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx\\ =\ &\lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}\frac{1}{1-q(1-H)}\int_{c\varepsilon}^\infty \frac{1}{y}\Phi_{Z^{(2)}}(y)dy = \lim_{\varepsilon\to 0}\frac{1}{1-q(1-H)}\Phi_{Z^{(2)}}(c\varepsilon) = \frac{1}{1-q(1-H)}\end{aligned}$$

where we used the fact that $\Phi_{Z^{(2)}}(y) \leq y^{-2}\mathbf{E}|Z^{(2)}|^2$ and so $\lim_{y\to\infty}\log y\Phi_{Z^{(2)}}(y) = 0$.

It remains to show that $\frac{1}{-\log c\varepsilon}\int_1^\infty P_1(x)d\left[\frac{1}{x}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]$ converges to zero as ε tends to 0 (note that actually it follows from a result by [1] that a Hermite random variable has a density, but we don't need it explicitly, we only use the fact that $\Phi_{Z^{(2)}}$ is differentiable almost everywhere). This is equal to

$$\begin{aligned}&\lim_{\varepsilon}\frac{1}{-\log c\varepsilon}\int_1^\infty P_1(x)c\varepsilon(1-q(1-H))x^{-q(1-H)-1}\Phi'_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx\\ =\ &c\frac{\varepsilon}{-\log\varepsilon}(c\varepsilon)^{\frac{q(1-H)}{1-q(1-H)}}\int_{c\varepsilon}^\infty P_1\left(\left(\frac{y}{c\varepsilon}\right)^{\frac{1}{1-q(1-H)}}\right)\Phi'_{Z^{(2)}}(y)y^{-\frac{1}{1-q(1-H)}}dy\\ \leq\ &c\frac{1}{-\log\varepsilon}\int_{c\varepsilon}^\infty P_1\left(\left(\frac{1}{c\varepsilon}\right)^{\frac{1}{1-q(1-H)}}\right)\Phi'_{Z^{(2)}}(y)dy\end{aligned}$$

which clearly goes to zero since P_1 is bounded and $\int_0^\infty \Phi'_{Z^{(2)}}(y)dy = 1$. ■

The next result estimates the limit of the difference between the functions $f_i(\varepsilon)$ given by (14), (15) and the sequence in Lemma 1.

Proposition 1 *Let $q \geq 2$ and $c > 0$.*

i. If $H < 1 - \frac{1}{2q}$, let $Z_n^{(1)}$ be given by (13) and let $Z^{(1)}$ be standard normal random variable. Then it holds

$$\frac{1}{-\log c\varepsilon}\left[\sum_{n\geq 1}\frac{1}{n}P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - \sum_{n\geq 1}\frac{1}{n}P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \to_{\varepsilon\to 0} 0.$$

ii. Let $Z^{(2)}$ be a Hermite random variable of order $q \geq 2$ and $H > 1 - \frac{1}{2q}$. Then

$$\frac{1}{-\log c\varepsilon}\left[\sum_{n\geq 1}\frac{1}{n}P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - \sum_{n\geq 1}\frac{1}{n}P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \to_{\varepsilon\to 0} 0.$$

Proof: Let us start with the point i. Assume $H < 1 - \frac{1}{2q}$. We can write

$$\begin{aligned}
&\sum_{n\geq 1}\frac{1}{n}P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - \sum_{n\geq 1}\frac{1}{n}P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\\
=\ &\sum_{n\geq 1}\frac{1}{n}\left[P\left(Z_n^{(1)} > c\varepsilon\sqrt{n}\right) - P\left(Z^{(1)} > c\varepsilon\sqrt{n}\right)\right] + \sum_{n\geq 1}\left[\frac{1}{n}P\left(Z_n^{(1)} < -c\varepsilon\sqrt{n}\right) - P\left(Z^{(1)} < -c\varepsilon\sqrt{n}\right)\right]\\
\leq\ & 2\sum_{n\geq 1}\frac{1}{n}\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)} > x\right) - P\left(Z^{(1)} > x\right)\right|.
\end{aligned}$$

It follows from [10], Theorem 4.1 that

$$\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)} > x\right) - P\left(Z^{(1)} > x\right)\right| \leq c\begin{cases}\frac{1}{\sqrt{n}}, & H \in (0, \frac{1}{2}]\\ n^{H-1}, & H \in [\frac{1}{2}, \frac{2q-3}{2q-2})\\ n^{qH-q+\frac{1}{2}}, & H \in [\frac{2q-3}{2q-2}, 1 - \frac{1}{2q}).\end{cases} \tag{16}$$

and this implies that

$$\sum_{n\geq 1}\frac{1}{n}\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(i)} > x\right) - P\left(Z^{(i)} > x\right)\right| \leq c\begin{cases}\sum_{n\geq 1}\frac{1}{n\sqrt{n}}, & H \in (0, \frac{1}{2}]\\ \sum_{n\geq 1} n^{H-2}, & H \in [\frac{1}{2}, \frac{2q-3}{2q-2})\\ \sum_{n\geq 1} n^{qH-q-\frac{1}{2}}, & H \in [\frac{2q-3}{2q-2}, 1 - \frac{1}{2q}).\end{cases} \tag{17}$$

and the last sums are finite (for the last one we use $H < 1 - \frac{1}{2q}$). The conclusion follows.

Concerning the point ii. (the case $H > 1 - \frac{1}{2q}$), by using a result in Proposition 3.1 of [2] we have

$$\sup_{x\in\mathbb{R}} \left| P\left(Z_n^{(i)} > x\right) - P\left(Z^{(i)} > x\right) \right| \leq c \left(\mathbf{E} \left| Z_n^{(2)} - Z^{(2)} \right|^2 \right)^{\frac{1}{2q}} \leq c n^{1-\frac{1}{2q}-H} \tag{18}$$

and as a consequence

$$\sum_{n\geq 1} \frac{1}{n} P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) - \sum_{n\geq 1} \frac{1}{n} P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) \leq c \sum_{n\geq 1} n^{-\frac{1}{2q}-H}$$

and the above series is convergent because $H > 1 - \frac{1}{2q}$. ■

We state now the Spitzer's theorem for the variations of the fractional Brownian motion.

Theorem 2 *Let f_1, f_2 be given by (14), (15) and the constants $c_{1,q,H}, c_{2,q,H}$ be those from Theorem 1.*

i. If $0 < H < 1 - \frac{1}{2q}$ then

$$\lim_{\varepsilon\to 0} \frac{1}{\log(c_{1,H,q}^{-1}\varepsilon)} f_1(\varepsilon) = 2.$$

ii. If $1 > H > 1 - \frac{1}{2q}$ then

$$\lim_{\varepsilon\to 0} \frac{1}{\log(c_{2,H,q}^{-1}\varepsilon)} f_2(\varepsilon) = \frac{1}{1 - q(1-H)}.$$

Proof: It is a consequence of Lemma 1 and Proposition 1. ■

Remark 2 *Concerning the case $H = 1 - \frac{1}{2q}$, note that the correct normalization of V_n (3) is $\frac{1}{(\log n)\sqrt{n}}$. Because of the appearance of the term $\log n$ our approach is not directly applicable to this case.*

4 Hsu-Robbins theorem for the variations of fractional Brownian motion

In this section we prove a version of the Hsu-Robbins theorem for the variations of the fractional Brownian motion. Concretely, we denote here by, for every $\varepsilon > 0$

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n\right) \tag{19}$$

if $H < 1 - \frac{1}{2q}$ and by

$$g_2(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n^{2-2q(1-H)}\right) \tag{20}$$

if $H > 1 - \frac{1}{2q}$. and we estimate the behavior of the functions $g_i(\varepsilon)$ as $\varepsilon \to 0$. Note that we can write

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right), \quad g_2(\varepsilon) = \sum_{n\geq 1} P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right)$$

with $Z_n^{(1)}, Z_n^{(2)}$ given by (13).

We decompose it as: for $H < 1 - \frac{1}{2q}$

$$\begin{aligned} g_1(\varepsilon) &= \sum_{n\geq 1} P\left(|Z^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) \\ &+ \sum_{n\geq 1}\left[P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right)\right]. \end{aligned}$$

and for $H > 1 - \frac{1}{2q}$

$$\begin{aligned} g_2(\varepsilon) &= \sum_{n\geq 1} P\left(|Z^{(2)}| > \varepsilon c_{2,q,H}^{-1} n^{1-q(1-H)}\right) \\ &+ \sum_{n\geq 1}\left[P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right)\right]. \end{aligned}$$

We start again by consider the situation when $Z_n^{(i)}$ are replaced by their limits.

Lemma 2 *i. Let $Z^{(1)}$ be a standard normal random variable. Then*

$$\lim_{\varepsilon\to 0}(c\varepsilon)^2 \sum_{n\geq 1} P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right) = 1.$$

ii. Let $Z^{(2)}$ be a Hermite random variable with $H > 1 - \frac{1}{2q}$. Then

$$\lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n\geq 1} P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) = \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}.$$

Proof: The part i. is a consequence of the result of Heyde [5]. Indeed take $X_i \sim N(0,1)$ in (1). Concerning part ii. we can write

$$\begin{aligned} &\lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n\geq 1} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \\ = &\lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}}\left[\int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx - \int_1^\infty P_1(x)d\left[\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]\right] \\ &:= \lim_{\varepsilon\to 0}(A(\varepsilon) + B(\varepsilon)) \end{aligned}$$

with $P_1(x) = [x] - x + \frac{1}{2}$. Moreover

$$\begin{aligned} A(\varepsilon) &= (c\varepsilon)^{\frac{1}{1-q(1-H)}} \int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx \\ &= \frac{1}{1-q(1-H)} \int_{c\varepsilon}^\infty \Phi_{Z^{(2)}}(y) y^{\frac{1}{1-q(1-H)}-1} dy. \end{aligned}$$

Since $\Phi_{Z^{(2)}}(y) \le y^{-2}$ we have $\Phi_{Z^{(2)}}(y) y^{\frac{1}{1-q(1-H)}} \to_{y\to\infty} 0$ and therefore

$$A(\varepsilon) = -\Phi_{Z^{(2)}}(c\varepsilon)(c\varepsilon)^{\frac{1}{1-q(1-H)}} - \int_{c\varepsilon}^\infty \Phi'_{Z^{(2)}}(y) y^{\frac{1}{1-q(1-H)}} dy$$

where the first terms goes to zero and the second to $\mathbf{E}\left|Z^{(2)}\right|^{\frac{1}{1-q(1-H)}}$. The proof that the term $B(\varepsilon)$ converges to zero is similar to the proof of Lemma 2, point ii. ■

Remark 3 *The Hermite random variable has moments of all orders (in particular the moment of order $\frac{1}{1-q(1-H)}$ exists) since it is the value at time 1 of a selfsimilar process with stationary increments.*

Proposition 2 *i. Let $H < 1 - \frac{1}{2q}$ and let $Z_n^{(1)}$ be given by (13). Let also $Z^{(1)}$ be a standard normal random variable. Then*

$$(c\varepsilon)^2 \sum_{n\ge1} \left[P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \to_{\varepsilon\to0} 0$$

ii. Let $H > 1 - \frac{1}{2q}$ and let $Z_n^{(2)}$ be given by (13). Let $Z^{(2)}$ be a Hermite random variable. Then

$$(c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n\ge1} \left[P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \to_{\varepsilon\to0} 0.$$

Remark 4 *Note that the bounds (16), (18) does not help here because the series that appear after their use are not convergent.*

Proof of Proposition 2: *Case $H < 1 - \frac{1}{2q}$.* We have, for some $\beta > 0$ to be chosen later,

$$\begin{aligned} &\varepsilon^2 \sum_{n\ge1} \left[P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \\ =\; &\varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} \left[P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \\ &+\varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} \left[P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \\ :=\; & I_1(\varepsilon) + J_1(\varepsilon). \end{aligned}$$

Consider first the situation when $H \in (0, \frac{1}{2}]$. Let us choose a real number β such that $2 < \beta < 4$. By using (16),

$$I_1(\varepsilon) \le c\varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} n^{-\frac{1}{2}} \le c\varepsilon^2 \varepsilon^{-\frac{\beta}{2}} \to_{\varepsilon \to 0} 0$$

since $\beta < 4$. Next, by using the bound for the tail probabilities of multiple integrals and since $\mathbf{E}\left|Z_n^{(1)}\right|^2$ converges to 1 as $n \to \infty$

$$J_1(\varepsilon) = \varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} P\left(Z_n^{(1)} > c\varepsilon\sqrt{n}\right) \le c\varepsilon^{-2} \sum_{n > [\varepsilon^{-\beta}]} \exp\left(\frac{-c\varepsilon\sqrt{n}}{\left(\mathbf{E}\left|Z_n^{(1)}\right|^2\right)^{\frac{1}{2}}}\right)^{\frac{2}{q}}$$

$$\le \varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} \exp\left(\left(-cn^{-\frac{1}{\beta}}\sqrt{n}\right)^{\frac{2}{q}}\right)$$

and since converges to zero for $\beta > 2$. The same argument shows that $\varepsilon^2 \sum_{n > [\varepsilon^{-\beta}]} P\left(Z^{(1)} > c\varepsilon\sqrt{n}\right)$ converges to zero.

The case when $H \in (\frac{1}{2}, \frac{2q-3}{2q-2})$ can be obtained by taking $2 < \beta < \frac{2}{H}$ (it is possible since $H < 1$) while in the case $H \in (\frac{2q-3}{2q-2}, 1 - \frac{1}{2q})$ we have to choose $2 < \beta < \frac{2}{qH - q + \frac{3}{2}}$ (which is possible because $H < 1 - \frac{1}{2q}$!).

Case $H > 1 - \frac{1}{2q}$. We have, with some suitable $\beta > 0$

$$\begin{aligned}
& \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n \ge 1} \left[P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \\
= \ & \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n=1}^{[\varepsilon^{-\beta}]} \left[P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \\
& + \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n \ge [\varepsilon^{-\beta}]} \left[P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \\
:= \ & I_2(\varepsilon) + J_2(\varepsilon).
\end{aligned}$$

Choose $\frac{1}{1-q(1-H)} < \beta < \frac{1}{(1-q(1-H))(2-H-\frac{1}{2q})}$ (again, this is always possible when $H > 1 - \frac{1}{2q}$!). Then

$$I_2(\varepsilon) \le c e^{\frac{1}{1-q(1-H)}} \varepsilon^{(-\beta)(2-H-\frac{1}{2q})} \to_{\varepsilon \to 0} 0$$

and by (9)

$$J_2(\varepsilon) \leq c \sum_{n>[\varepsilon^{-\beta}]} \exp\left(\left(\frac{-c\varepsilon n^{1-q(1-H)}}{\left(\mathbf{E}\left|Z_n^{(2)}\right|^2\right)^{\frac{1}{2}}}\right)^{\frac{2}{q}}\right) \leq c \sum_{n>[\varepsilon^{-\beta}]} \exp\left(cn^{-\frac{1}{\beta}} n^{1-q(1-H)}\right)^{\frac{2}{q}} \to_{\varepsilon\to 0} 0$$

$\blacksquare$

We state the main result of this section which is a consequence of Lemma 2 and Proposition 2.

Theorem 3 *Let $q \geq 2$ and let $c_{1,q,H}, c_{2,q,H}$ be the constants from Theorem 1. Let $Z^{(1)}$ be a standard normal random variable, $Z^{(2)}$ a Hermite random variable of order $q \geq 2$ and let g_1, g_2 be given by (19) and (20). Then*

i. *If $0 < H < 1 - \frac{1}{2q}$, we have $(c_{1,q,H}^{-1}\varepsilon)^2 g_1(\varepsilon) \to_{\varepsilon\to 0} 1 = \mathbf{E}Z^{(1)}$.*

ii. *If $1 - \frac{1}{2q} < H < 1$ we have $(c_{2,q,H}^{-1}\varepsilon)^{\frac{1}{1-q(1-H)}} g_2(\varepsilon) \to_{\varepsilon\to 0} \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}$.*

Remark 5 *In the case $H = \frac{1}{2}$ we retrieve the result (1) of [5]. The case $q = 1$ is trivial, because in this case, since $V_n = B_n$ and $\mathbf{E}V_n^2 = n^{2H}$, we obtain the following (by applying Lemma 1 and 2 with $q = 1$)*

$$\frac{1}{\log\varepsilon} \sum_{n\geq 1} \frac{1}{n} P\left(|V_n| > \varepsilon n^{2H}\right) \to_{\varepsilon\to 0} \frac{1}{H}$$

and

$$\varepsilon^2 \sum_{n\geq 1} P\left(|V_n| > \varepsilon n^{2H}\right) \to_{\varepsilon\to 0} \mathbf{E}\left|Z^{(1)}\right|^{\frac{1}{H}}.$$

Remark 6 *Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of i.i.d. centered random variable with finite variance and let $(a_i)_{i\geq 1}$ a square summable real sequence. Define $X_n = \sum_{i\geq 1} a_i \varepsilon_{n-i}$. Then the sequence $S_N = \sum_{n=1}^{N} [K(X_n) - \mathbf{E}K(X_n)]$ satisfies a central limit theorem or a non-central limit theorem according to the properties of the measurable function K (see [6] or [14]). We think that our tools can be applied to investigate the tail probabilities of the sequence S_N in the spirit of [5] or [12] at least the in particular cases (for example, when ε_i represents the increment $W_{i+1} - W_i$ of a Wiener process because in this case ε_i can be written as a multiple integral of order one and X_n can be decomposed into a sum of multiple integrals. We thank the referee for mentioning the references [6] and [14].*

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